ACMS Problem Solving Seminar - Fall 2005
Problem Set 10 - Enumerative Combinatorics

Giovanni DiMatteo

Note: The following problems are taken from various sources, which are listed in pdf form on the ACMS problem solving seminar webpage.[1]

69. Let p be a prime integer. Prove that for any integer $1 < k < n$, p divides $\binom{p}{k}$. Prove that $(x+y)^p \equiv x^p + y^p (\text{mod } p)$ for all integers x and y.

70. Show that a square can be dissected by finitely many cuts and the resulting pieces can be rearranged into a regular octogon.

71. Let $p(\omega) = a_n\omega^n + a_{n-1}\omega^{n-1} + \ldots + a_1\omega + a_0$ be a polynomial function on the complex numbers with real coefficients, $a_n, \ldots, a_0 \in \mathbb{R}$. Show that if z is a complex root of p, then so is its conjugate $\overline{z}$.

72. A $10 \times 10 \times 10$ cube is formed of smaller unit cubes. A grasshopper sits at the center in one of the corner cubes C. At any given moment, it can jump from its present location into a cube which shares a common face with the cube where it sits, so long as the jump increases the distance between point C and the current position of the grasshopper. How many ways are there for the grasshopper to reach the unit cube at the opposite corner?

73. Let m, n, and k be positive integers such that $1 \leq k \leq n$. Use a counting argument to prove $\binom{m+n}{k} = \sum_{i=1}^{k} \binom{n}{i}\binom{m+n}{k-i}$.

74. Prove each of the following identities (preferably by a combinatorial argument, but any method will suffice).

    i. $\binom{n}{k} = \binom{n}{n-k}$

    ii. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

    iii. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

    iv. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

[1] Email GAD10@albion.edu for (non-spoiler) hints!

75. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}$$

76. Prove that for integers k and n, with $0 < k \leq n$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}$$