ACMS Problem Solving Seminar - Fall 2005
Problem Set 9 - Polynomials

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Note: The following problems are taken from various sources, which are listed in pdf form on the ACMS problem solving seminar webpage.[1]

60. Do there exist polynomials p and q with integer coefficients such that $(x^2 + x + 1)p(x) + (4x - 1)q(x) = 1$ identically?

61. One day in Mathsland, the king was feeling generous, so he ordered a guard to go and open all 100 locks of the prison. However, he changed his mind and then sent a guard to go and turn the locks back on every second cell. He changed his mind again, sending a guard to turn the lock on every third cell, and so on until he finally sent a guard to just turn the 100th lock. Who gets to go free?

62. The numbers $+1$ and -1 are placed in some entries of a 50×50 array, and it is known that the absolute of their sum is not greater than 100. Prove there is a 25×25 sub array such that the absolute value of the sum of entries is not greater than 25.

63. The $\mathfrak{Wi}^{\mathfrak{cked}}_{\mathfrak{tch}}$ boiled up a polynomial $p(x)$ with nonnegative integer coefficients. She has taken young Hans hostage and is getting ready to bake him into a pie. Hans can ask for a value of $p(x)$ and another value of $p(x)$ after that; if he guesses the polynomial correctly, he is freed. Will Hans escape?

64. Find all solutions to $(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x$.

65. Solve the system $m + n + k = 14$, $mn + nk + km = 33$, $mnk = 84$ in the integers or show it has no solutions.

66. Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

[1] Email GAD10@albion.edu for (non-spoiler) hints!

67. Let $P(x)$ be a polynomial of degree n, for which $P(x) \geq 0$ for all real numbers x. Prove that

$$P(x) + P'(x) + P''(x) + \ldots + P^{(n)}(X) \geq 0$$

for all real numbers x.

68. Let $az^2 + bz + c$ be a polynomial with complex coefficients such that a and b are nonzero. Prove that the zeroes of this polnomial lie in the region

$$|z| \leq |\frac{b}{a}| + |\frac{c}{b}|.$$