ACMS Problem Solving Seminar - Spring 2006
Calculus: Problem Set I - Sums and Series

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Note: As usual, the second and third problems are chosen to be unrelated to this week's topic. A list of sources for each problem will be posted on the ACMS website at the end of the semester or upon request.

1. Prove *Catalan's Identity*:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

2. Numbers are placed on the vertices of a 100-gon in such a way that each of them equals the arithmetic mean of its neighbors. Prove that all of the numbers are equal.

3. A dragon has 100 heads. A knight can cut off $15, 17, 20$, or 5 heads, respectively, with one blow of his sword. In each of these cases, $24, 2, 14$, or 17 new heads grow on its shoulders. If all heads are blown off, the dragon dies. Can the dragon ever die?

4. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

5. Determine the value of each of the following sums:

$$\sum_{k=1}^{n} k^2 \binom{n}{k}, \qquad \sum_{k=1}^{n} \frac{1}{k+1} \binom{n}{k}, \qquad \sum_{k=1}^{n} k^2 2^{-k}$$

6. If n is a positive integer, then $\sigma(n)$ denotes the sum of all divisors of n, and is an example of an *arithmetic function.*

   i. Show that if $n = uv$, where u and v are relatively prime[1], then $\sigma(n) = \sigma(u)\sigma(v)$; in this situation, σ is said to be *partially multiplicative.* Find an example of u and v for which this equality doesn't hold.

[1] Integers u and v are said to be *relatively prime* if they have no prime divisor in common

ii. Let p be prime. Find a closed form for $\sigma(p^m)$, where m is a positive integer.

iii. Use parts i. and ii. to find a closed form for $\sigma(n)$ for an arbitrary positive integer n.

7. Modify the harmonic sum

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

by removing all terms for which k contains a 9 in its decimal expansion. Prove that the resulting series converges.

8. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n < a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{k=1}^{\infty} a_k$ diverges.

9. Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression with common difference d.[2] Compute

$$\sum_{k=1}^{n} \frac{1}{a_k a_{k+1}}$$

[2]An *arithmetic progression* with common difference d is a set of n numbers, $a_1 + 0, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots, a_1 + (n-1)d$.