ACMS Problem Solving Seminar - Fall 2005
Problem Set 8 - Complex Numbers

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Note: The following problems are taken from various sources, which are listed in pdf form on the ACMS problem solving seminar webpage.[1]

53. Show that for a complex number $z = z_1 + z_2 \cdot i$,

$$2^{-1/2}(|z_1| + |z_2|) \leq |z| \leq |z_1| + |z_2|$$

54. Given three squares with side lengths 2, 3, and 6, perform only two cuts and reassemble the resulting five pieces into a square having side length equal to 7.

55. Let r_1, s_1, r_2, s_2 be rational numbers such that

$$r_1 + s_1\sqrt{3} = r_2 + s_2\sqrt{3}$$

Prove that $r_1 = r_2$ and $s_1 = s_2$.

56. Let z be a complex number such that $z + \frac{1}{z} = 2\cos(\theta)$ for some fixed real number θ. Prove that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ for all positive integers n.

57. Let H denote the set of positive integers that can be written as a sum of two squares. Prove that H is closed under multiplication.

58. Determine the sum of binomial coefficients

$$\sum_{k=0}(-1)^k \binom{n}{2k} \qquad \text{and} \qquad \sum_{k=0}\binom{n}{3k}$$

59. Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots$, then R_n. For an arbitrary point (x, y), find and simplify the coordinates of $R(x, y)$.

[1] Email GAD10@albion.edu for (non-spoiler) hints!