ACMS Problem Solving Seminar - Fall 2005
Problem Set 7 - Arithmetic

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Note: The following problems are taken from various sources, which are listed in pdf form on the ACMS problem solving seminar webpage.[1]

44. Determine whether the diophantine equation $x^2 - 3y^2 = 17$ has solutions in the integers.

45. Show that for all real numbers a, b, x, y, we have $2abxy \leq a^2x^2 + b^2y^2$.

46. The kingdom of Mathsland has several cities. One day, the king exiles an obnoxious citizen from the capitol to the city farthest from it. However, the citizen is especially obnoxious and is exiled once again to the city farthest from the current location, which happens to be different from the capitol. Prove that if our citizen continues to be perpetually exiled in this way, he'll never return to the capitol.

47. Find, with proof, the largest positive integer that cannot be written as a sum of two composite integers.

48. Let n be a positive integer s the sum of its digits. Prove That $n \equiv s \pmod 3$. This shows that 3 divides n if and only if 3 divides the sum of its digits; you can now easily determine whether 4637284909057162499356121517100111 is divisible by 3.

49. Solve the system $m + n + k = 31$, $mn + nk + km = 43$, $mnk = 27$ in the integers or show it has no solutions.

50. The $\mathfrak{Wi}^{\mathfrak{cked}}_{\mathfrak{tch}}$ says that she knows a positive integer n that, upon subtracting each of its digits from the number, gives 1234567. Prove that she is a liar.

51. Prove that $\frac{\gcd(n,k)}{n}\binom{n}{k}$ is an integer for all pairs of integers $n \geq k \geq 1$. [2]

52. Construct an infinite set of positive integers having the following properties: any two elements of the set are not relatively prime, no integer greater than 1 divides all elements of the set, no element of the set divides any other element.

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[1] Email GAD10@albion.edu for (non-spoiler) hints!

[2] Here, $\binom{n}{k}$ denotes n *choose* k, the number of ways to pick a subset of size k from a set of size n. Convince yourself that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.