ACMS Problem Solving Seminar - Spring 2006
Calculus: Presentation I - Sums and Series
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Finite Sums

Some Examples

1. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
2. $1 + x + x^2 + \ldots + x^{n-1} = \frac{1-x^n}{1-x}$
3. $\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^k$
4. $\lfloor \frac{a}{b} \rfloor + \lfloor \frac{2a}{b} \rfloor + \lfloor \frac{3a}{b} \rfloor + \ldots + \lfloor \frac{(b-1)a}{b} \rfloor$

Summation by Parts:

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two finite sequences of numbers. Then

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = (a_1 - a_2)b_1 + (a_2 - a_3)(b_1 + b_2) + (a_3 - a_4)(b_1 + b_2 + b_3) + \ldots$$
$$+(a_{n-1} - a_n)(b_1 + b_2 + \ldots + b_{n-1}) + a_n(b_1 + b_2 + \ldots + b_n)$$

Some Properties of Finite Sums:

i. $\sum_{i=0}^n \sum_{j=0}^n a_i a_j = \sum_{j=0}^n \sum_{i=0}^n a_i a_j$

ii. $\sum_{j=0}^n \sum_{i=j}^n a_i a_j = \sum_{i=0}^n \sum_{j=i}^n a_i a_j$

iii. $\left(\sum_{i=0}^n a_i\right)^2 = \sum_{i=0}^n \sum_{j=0}^n a_i a_j = 2\left(\sum_{i=0}^n \sum_{j=0}^i a_i a_j\right) - \sum_{i=0}^n a_i^2$

Infinite Sums

Definition: If $(a_i)_{i=1}^{\infty}$ is an infinite list of (complex) numbers, then the *Nth partial sum*

$$S_N = \sum_{i=0}^{N} a_i$$

is a finite sum, and hence defined. Formally, the *infinite series* is defined as

$$\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} S_N$$

where the limit of the sequence $(S_N)_{n=1}^{\infty}$ is the number S so that for every small $\epsilon > 0$, there is an N so that $|S - S_N| < \epsilon$. We say that the sum *converges* if S exists. If not, then the sum *diverges*. We will revisit the notion of a limit formally in several weeks.

Some Properties of Infinite Sums:

i. If a $\sum a_i$ is convergent, it is necessarry that $\lim_{n\to\infty} a_n = 0$.

ii. If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges. Also, if $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

iii. **Caution!** While rearranging the terms in a finite sum does not change the value of that sum, the same cannot be said for infinite sums. It is clear that one can choose a finite collection of terms of a sum and rearrange those (why?), but one may not generally rearrange infinitely many terms in the sum. To illustrate this, sum the numbers in the array below in two different ways:

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

However, if the sum converges absolutely, then rearrangement of an infinite sum can be done without changing its value; notice that the sum of the array above does not converge absolutely.

Some Examples

1. $\frac{3}{1\cdot2\cdot3} + \frac{5}{2\cdot3\cdot4} + \frac{7}{3\cdot4\cdot5} + \frac{9}{4\cdot5\cdot6} + \ldots$
2. $e := \sum_{k=0}^{\infty} \frac{1}{k!}$; *Euler's number* is irrational.
3. $\sum_{k=1}^{\infty} \frac{1}{k}$; *The Harmonic Series* diverges.
4. The *Riemann zeta function* is defined as $\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{n^s}$ for $s > 1$.

Sources and Further Reading:

[1.] Andreescu, Titu, and Răzvan Gelca. Mathematical Olympiad Challenges. Boston: Birkhäuser, 2003.

[2.] Knopp, Konrad. Theory and Application of Infinite Series. London: Blackie & Son Limited, 1928.

[3.] Larson, Loren C. *Problem-Solving Through Problems.* New York: Springer, 1983.

[4.] Rudin, Walter. Principles of Mathematical Analysis. 3rd ed. New York: McGraw-Hill, 1976.