

# ACMS Problem Solving Seminar - Fall 2005

## Presentation Material 8 - Complex Numbers

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**Note:** The following problems are taken from various sources, which are listed in pdf form on the ACMS problem solving seminar webpage.

This week, we will examine problems that have solutions relying on basic facts about complex numbers. The following facts about complex numbers should be kept in mind:

**The Field of Complex Numbers:** We define the *complex numbers* to be the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, \text{ and } i^2 + 1 = 0\}$$

together with addition and multiplication operations  $+$  and  $\cdot$  defined as follows: for complex numbers  $z = z_1 + z_2i$  and  $w = w_1 + w_2i$ ,

$$z + w = (z_1 + w_1) + (z_2 + w_2)i \quad \text{and} \quad z \cdot w = (z_1w_1 - z_2w_2) + (z_1w_2 + z_2w_1)i$$

These definitions are natural enough. For example, the definition of multiplication comes from assuming the field properties and using the relation  $i^2 + 1 = 0$ . With these operations, it's possible to show that  $(\mathbb{C}, +, \cdot)$  is a field.

Complex numbers are said to have *real* and *imaginary* parts. If  $z = z_1 + z_2i$ , then the real part of  $z$  is denoted by  $\Re(z) = z_1$  and  $\Im(z) = z_2$  denotes the imaginary part of  $z$ . The *conjugate* of  $z$  is  $\bar{z} = z_1 - z_2i$ . The *norm* of a complex number is defined as  $N(z) = \sqrt{z_1^2 + z_2^2}$

**Proposition:** The following are easy to show from the definitions above:

1.  $\overline{\bar{z}} = z$
2.  $z + \bar{z} = 2 \cdot \Re(z)$
3.  $z - \bar{z} = 2 \cdot \Im(z) \cdot i$
4.  $z \cdot \bar{z} = N(z)^2$
5.  $N(z \cdot w) = N(z)N(w)$

**The Geometric Interpretation of  $\mathbb{C}$ :** Each of the properties of complex numbers mentioned above have a geometric interpretation. To a complex number  $z = z_1 + z_2 \cdot i$ , we associate the point  $(z_1, z_2)$  in the plane. We can use trigonometry to work in an equivalent frame. The complex number  $z$  can also be written in the form

$$z = r(\cos(\theta) + i \cdot \sin(\theta))$$

where  $r = N(z)$  and  $\theta = \arg(z)$  (the *argument* of  $z$ , the angle that  $z$  makes with the horizontal axis of the complex plane). The operations themselves have natural geometric interpretations. Addition of complex numbers  $z$  and  $w$  can be realized by taking  $z$  and  $w$  to be legs of a parallelogram with a vertex at the origin of the complex plane, and the sum of  $z$  and  $w$  to be the vertex of the parallelogram opposite to the origin. Multiplication has a slightly more interesting interpretation. The product of complex numbers  $z$  and  $w$  is just the complex number  $u$  with  $N(u) = N(z)N(w)$  and  $\arg(u) = \arg(z) + \arg(w)$ . You can prove this using the trigonometric representation of complex numbers.

**The  $n^{\text{th}}$  roots of unity:** A complex solution to the polynomial relation  $x^n - 1 = 0$  is called an  $n^{\text{th}}$  root of unity. If we put

$$\zeta_n = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$$

then the numbers  $1 = \zeta_n^0, \zeta_n^1, \zeta_n^2, \dots, \zeta_n^{n-1}$  are all  $n$  of the  $n^{\text{th}}$  roots of unity.  $\zeta_n$  is called a *primitive  $n^{\text{th}}$  root of unity*.

**DeMoivre's Theorem:** The above can be proven using DeMoivre's theorem: If  $z = r[\cos(\theta) + i \sin(\theta)]$ , then

$$z^n = r^n[\cos(n\theta) + i \sin(n\theta)]$$

**The Ring of Gauss Integers:** The set  $\mathbb{Z}[i] = \{u + vi | u, v \in \mathbb{Z}\}$  forms a *ring* when we apply the complex operations to it. A *ring* is a structure in which every property of fields holds except for inverses. In particular, the product of two Gauss integers is a Gauss integer, and so is the sum.

**Some Problems:** complex numbers are highly applicable; they show up in applied mathematics, differential equations, number theory, physics, linear algebra, and Galois theory (to name a few disparate areas). It turns out they are pretty nice for solving some elementary problems. The facts collected in this handout should be sufficient for learning how to use complex numbers in elementary problem solving.

1. Find all values of  $i^n$  using the relation  $i^2 + 1 = 0$ .
2. Show that every complex number is a root of some polynomial with real coefficients.
3. Suppose we have tiled a  $m \times n$  rectangle with  $1 \times r$  dominoes. Prove that  $r|m$  or  $r|n$ .