ACMS Problem Solving Seminar - Fall 2005
Presentation Material 7 - Arithmetic
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Note: The following problems are taken from various sources, which are listed in pdf form on the ACMS problem solving seminar webpage.

This week, we will examine problems that have solutions relying on basic facts about the integers. The following facts about integers should be kept in mind:

Division With Remainder: Let n and k be integers with and $k > 0$. Then *there exist* unique integers $q, r \geq 0$ with $0 \leq r < k$ such that

$$n = qk + r.$$

The integers q and r are uniquely determined by n and k. In this situation, q is called the *quotient* and r is called the *remainder*.

Base d representation: Let $d \geq 2$ be a positive integer. Then every positive integer N can be written uniquely in the form

$$N = a_k d^k + \ldots + a_2 d^2 + a_1 d + a_0$$

where $0 \leq a_i < d$ for all $i = 0, 1, \ldots, k$.

The Distinguished Common Divisor: Given m, n (not both zero), then their *distinguished common divisor* d is the unique integer defined by

1. $d > 0$
2. $d|m$ and $d|n$
3. If $k|m$ and $k|n$, then $k|d$

We write, $d = \mathrm{dcd}(m, n)$.[1] We define the dcd for a finite collection of integers in a similar manner.

Proposition: There are integers x and y such that $\mathrm{dcd}(m, n) = mx + ny$.

Definition: A positive integer $p \geq 2$ is said to be *prime* if $p|mn$ implies $p|m$ or $p|n$.

Proposition: Every positive integer can be written uniquely as a product of primes.

[1] Some people call this the *distinguished common divisor* of m and n.

Some Problems:

1. The king of mathsland was in a bad mood; he summoned his son-in-law and wrote down three secret two-digit numbers a, b, and c and told the son-in-law that he could choose three numbers X, Y, and Z, then the king would tell him $aX + bY + cZ$. The king's son-in-law must identify a, b, and c, or else he will be locked in a dungeon. Help him out of this dangerous situation.

2. If an integer has base-10 coefficients

$$M = a_n a_{n-1} \dots a_1 a_0$$

show that $M \equiv (a_0 - a_1 + a_2 - \dots (-1)^n a_n) \pmod{11}$

3. Let x and y be positive integers such that $x + y = p$ for some prime p. Then $\mathrm{dcd}(x, y) = 1$.

4. Determine whether the Diophantine equation

$$6x^2 + 5y^3 = 8$$

has solutions in integers x and y.