ACMS Problem Solving Seminar - Fall 2005
Presentation Material 6 - Combinatorial Games
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Note: The following problems are taken from various sources, which are listed in pdf form on the ACMS problem solving seminar webpage.

What is a "game"? We have been playing games since we were children; almost everyone in our culture is familiar with tic-tac-toe, chess, checkers, dominoes, scrabble, and other things commonly called "games." Games can be made out of almost any common thing, as we've seen in our problem sets, and sometimes the games are really not games–someone can win quite easily without having much experience or "skill" at all. Some games could be random; if you and I flip a coin, say I win if it lands heads while you win if it lands tails. In some sense, a coin-flipping game is not a good game...

Definition: By a *combinatorial game*, we will mean a game for two players in which the following holds:

1. The game consists of a set of well-defined "moves," and the two players take turns making these "moves."

2. The game is *finite*; that is, it is over in a finite number of moves, no matter which moves the players make.

3. The game has a condition, called "win," that both players try to satisfy by making moves and after which no moves are allowed. If a player does not attain this condition, we call him/her a *loser*. Another way to put this is that a player who makes the last allowed move wins.[1]

Zermelo's Theorem: Essentially shows that in a given combinatorial game, one of the players has a winning strategy. That is, one of the two players, player one or two, can always win if s/he plays the game to win.[2]

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[1]This is not rigorous, but will suffice for our purposes.
[2]The above caveat applies here as well.

Some Examples:

1. Two players take turns placing rooks (castles) on a chessboard so that they cannot capture eachother. The loser is the player who cannot place a rook. Who will win?

2. There are two piles of 7 stones each. At each turn, a player may take as many stones as he chooses, but only from one of the piles. The loser is the player who cannot move.

The Divisor Game: This game is for two players. Choose a natural number N and name distinct, positive divisors of N by turns. The rules for moving are the following

i. No divisor of N that is a multiple of a previously pronounced number can be named.

ii. The player who is forced to name 1 loses the game. The other player is the winner.

Proposition: There is a winning strategy for the first player in the divisor game.

Proof: It's clear that the divisor game is a combinatorial game; the moves are well-defined, and the game is over in a finite number of steps (for example, the number of divisors of N is an upper bound for the number of moves made in the game). Suppose that the first player has no winning strategy. Then by Zermelo's theorem, the second player has a winning strategy, and no matter which divisor player one picks, player two will proceed along in an appropriate way and win. So, in particular, player 1 can pick N and watch what player two does. But if player two wins by first picking d, say, then player one could have picked d on the first move, and taken on the role that player 2 (the supposed winner) is currently in. Thus, player one has a winning strategy.

Proposition: $m \times n$ "Chomp" is equivalent to the divisor game when $N = p^m q^n$, where p and q are distinct primes. The rotton cherry corresponds to 1.